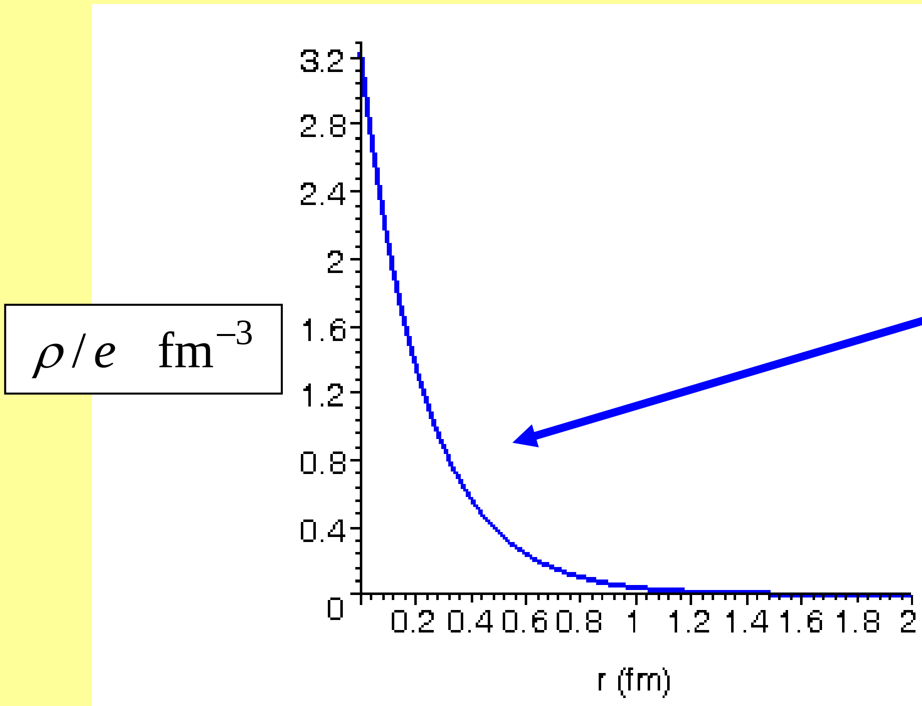
- What we know about the proton's internal structure comes from scattering experiments
- Experiments at SLAC (Stanford Linear Accelerator Centre) in the 1960's and 70's found that the proton has an extended electric charge distribution:



$$\rho(r) = e\rho_o \exp(-M\ r)$$

$$M = 4.33 \ \mathrm{fm}^{-1}$$

$$\left\langle r^2 \right\rangle^{1/2} = \frac{\sqrt{12}}{M} = 0.80 \ \mathrm{fm}$$

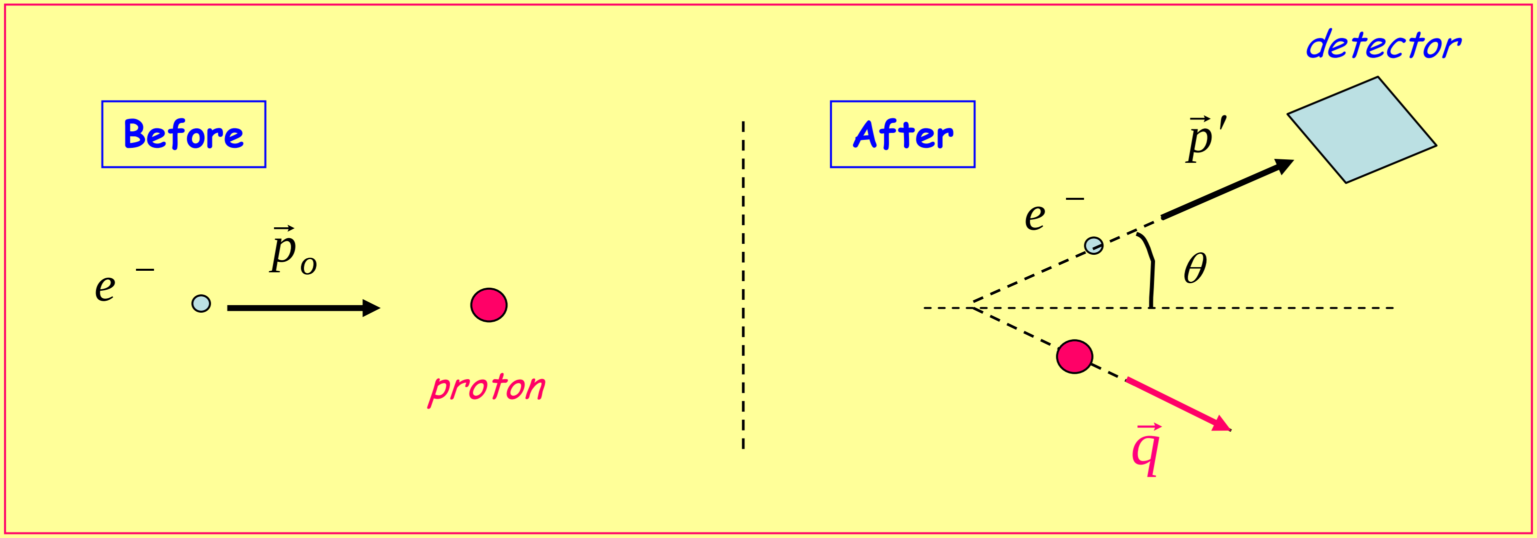
(Ref "Accelerators" – F&H sec. 2.1 – 2.4)

- Nobel prize, 1990 to Friedman, Kendall and Taylor (Cdn!) for deep inelastic scattering experiments that showed the existence of pointlike constituents inside the proton:

http://www.nobel.se/physics/laureates/1990/illpres/

Overview: Electron scattering experiments: *(Ref: F&H chapter 6)*

electron scatters from the proton's electric charge distribution $\rho(r)$



Scattering rate is proportional to the cross-section, which depends on the momentum transferred to the struck particle, *q:*

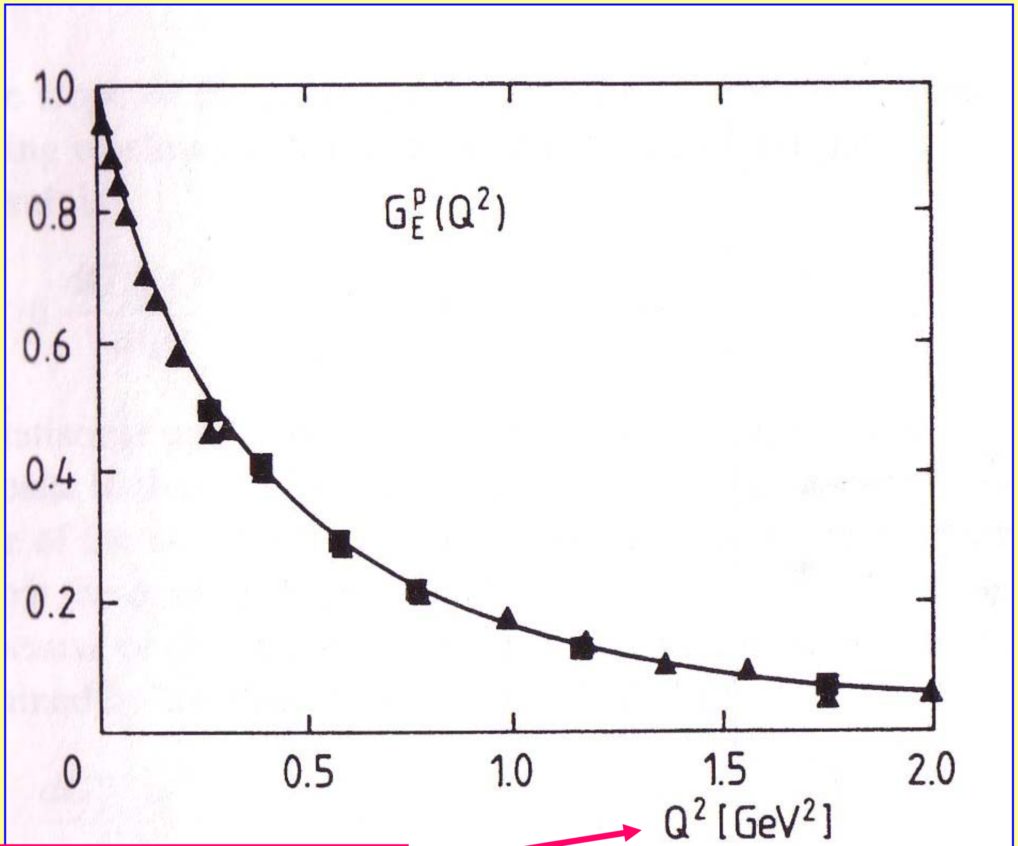
$$\frac{d\sigma}{d\Omega}(\theta) = \left.\frac{d\sigma}{d\Omega}\right)_o \left[F(q^2)\right]^2$$

point charge result (known)

Dimensionless "Form factor" $|F(q^2)| \leq 1$ gives the Fourier transform of the extended charge distribution

$$\left(\frac{\rho(r)}{e}\right) = \frac{1}{(2\pi)^3} \int e^{-i\vec{q}.\vec{r}}\, F(q^2)\, d^3q$$

What it looks like: proton electric form factor



4 – momentum transfer: Q^2
(more soon on this!)

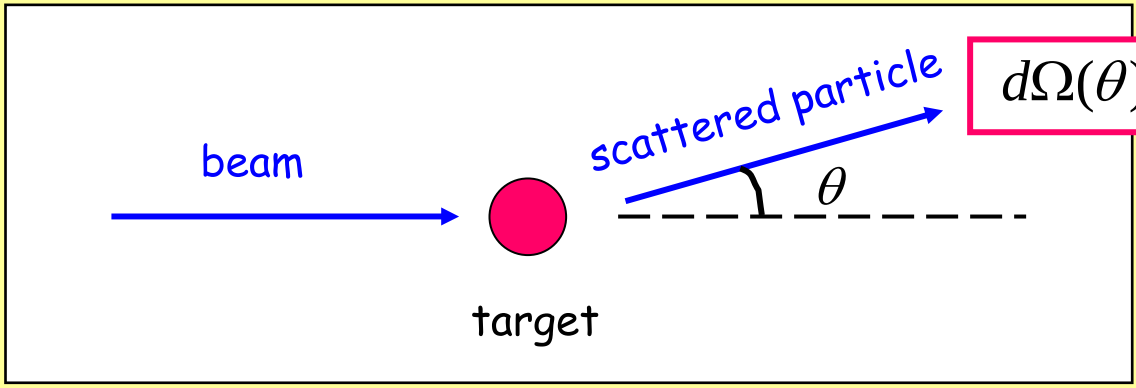
Ref: Arnold et al., Phys. Rev. Lett. 57, 174 (1986)

"Dipole formula"

$$G_E^p(Q^2) = \left(1 + \frac{Q^2}{0.71\,\mathrm{GeV}^2}\right)^{-2}$$

(Inverse Fourier transform gives charge density $\rho(r)$)

Details, Part I: Scattering Cross Section: σ, $d\sigma/d\Omega$ *(Ref: F&H section 6.2)*



$$d\Omega(\theta) = \sin\theta \, d\theta \, d\phi$$

- A beam particle will scatter from the target at any angle θ if it approaches within a (perpendicular) **cross sectional area** σ centered on the target particle.

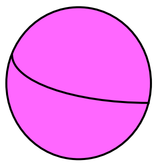
→ <u>Definition:</u> **total scattering cross section** σ (units: area, eg. fm^2)

- Scattering into a particular solid angle at (θ,ϕ) in 3d occurs if the beam particle approaches within a (perpendicular) cross sectional area $d\sigma/d\Omega$ centered on the target

→ <u>Definition:</u> **differential scattering cross section $d\sigma/d\Omega$** (units: area/solid angle)

$$\Rightarrow \quad \boxed{\int_{4\pi} \frac{d\sigma(\theta,\phi)}{d\Omega} \, d\Omega = \sigma}$$

Units, etc: What is the right scale for σ?



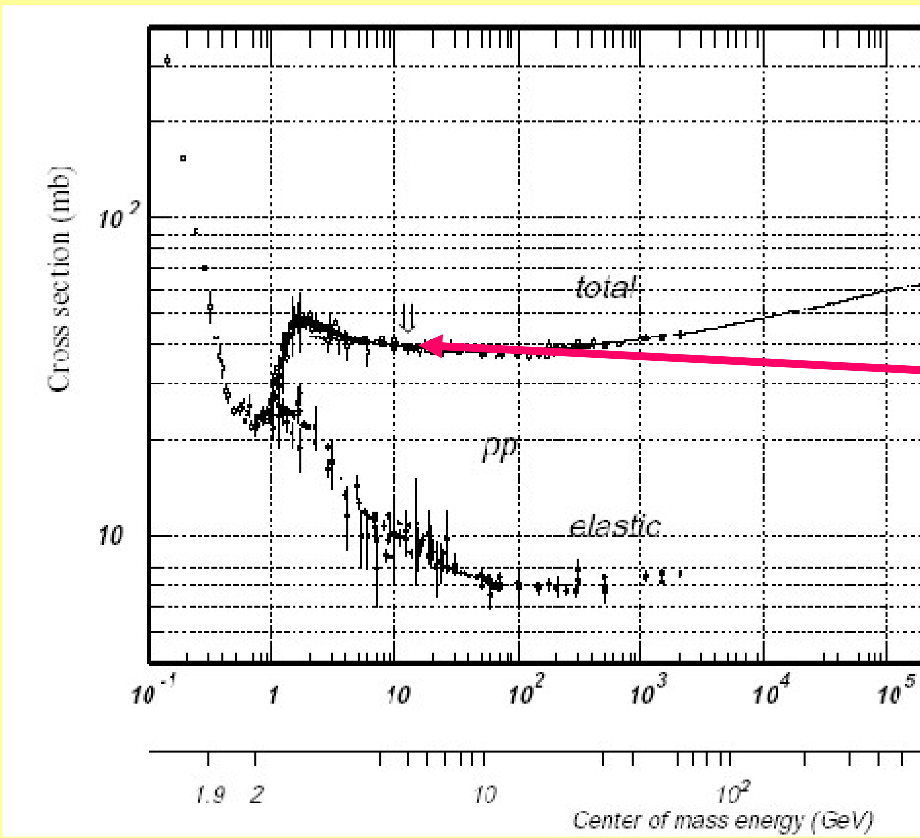
proton, R ~ 0.8 fm

Cross-sectional area:

$$A = \pi R^2 = 2.0 \quad \text{fm}^2$$

$$\boxed{\sigma = A\,?}$$

WRONG! Geometry has nothing to do with the value of σ. Scale is set by the interaction and beam energy



Scale for proton-proton scattering: ~ 0.01 b

Cross section unit: "barn"

1 barn = 10^{-24} cm^2 = 100 fm^2

σ: b   dσ/dΩ: b/sr

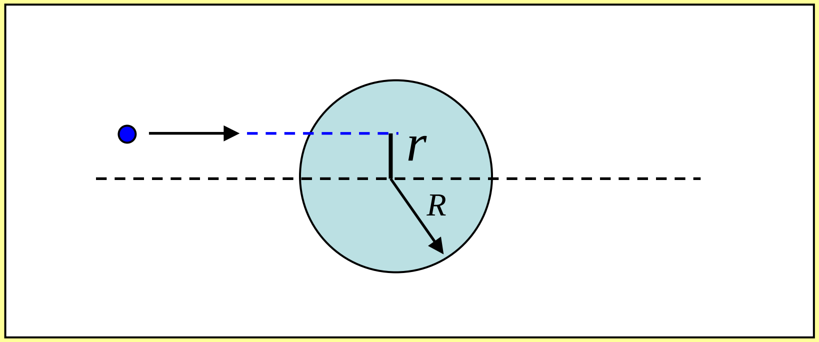
Other reactions:

ν-p: $\sigma \sim 10^{-14}$ b
e-p: $\sigma < 10^{-9}$ b

but energy-scale dependent!

Warmup exercise: classical / geometrical scattering process:

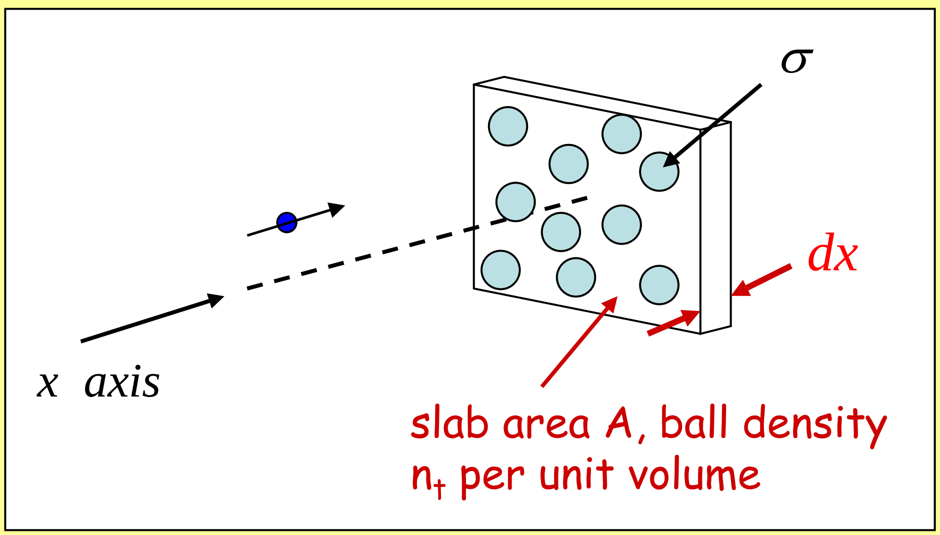
e.g. small bullet, fired at a billiard ball of radius R:



Scattering probability, P:

P = 1 if $r \leq R$ ("hit")
P = 0 if $r > R$ ("miss")

Next, consider an array of target balls fired on by a random stream of bullets, i.e. a "beam" of bullets whose cross sectional area is much larger than any given target ball:



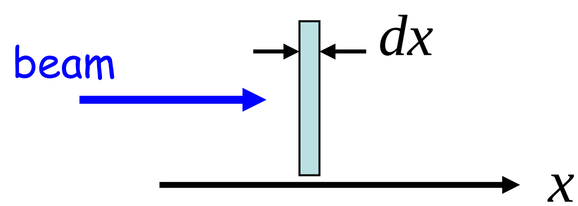
Scattering probability, P:

$$P = \frac{(\#\text{balls}) \times (\text{area per ball})}{\text{area of target}}$$

$$= \frac{(n_t \; A \, dx) \times \sigma}{A} = n_t \; \sigma \; dx$$

Connection to Experiment – same result, but σ is not the geometrical size

- Experimenters measure the **scattering rate** into a given solid angle $\Delta\Omega$ at (θ, ϕ).
- Knowing target thickness, detector efficiency and solid angle yields $d\sigma/d\Omega$

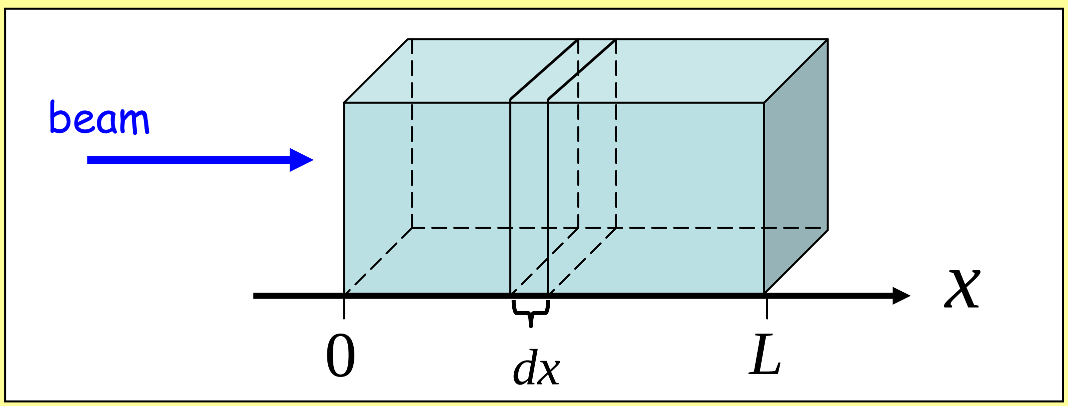


Interaction probability: $dP = n_t \; \sigma \; dx$

n_t = # of target nuclei per unit volume

Transmission: $T(x)$ = probability of getting to x without interacting = $1 - P(x)$

$$T(x+dx) = T(x)\,[1 - dP] = T(x)\,[1 - n_t\,\sigma\,dx]$$



$$\frac{dT}{T} = -n_t\,\sigma\,dx$$

$$T(x) = e^{-n_t\,\sigma\,x}$$

A target is said to be **"thin"** if the transmission probability is close to 1.

Then, for target thickness x :

$$\boxed{P(x) \;=\; (1 - T(x)) << 1 \;\Rightarrow\; P(x) \;\cong\; n_t\, \sigma\, x}$$

thin target:

$$\boxed{x << \frac{1}{n_t \sigma}}$$

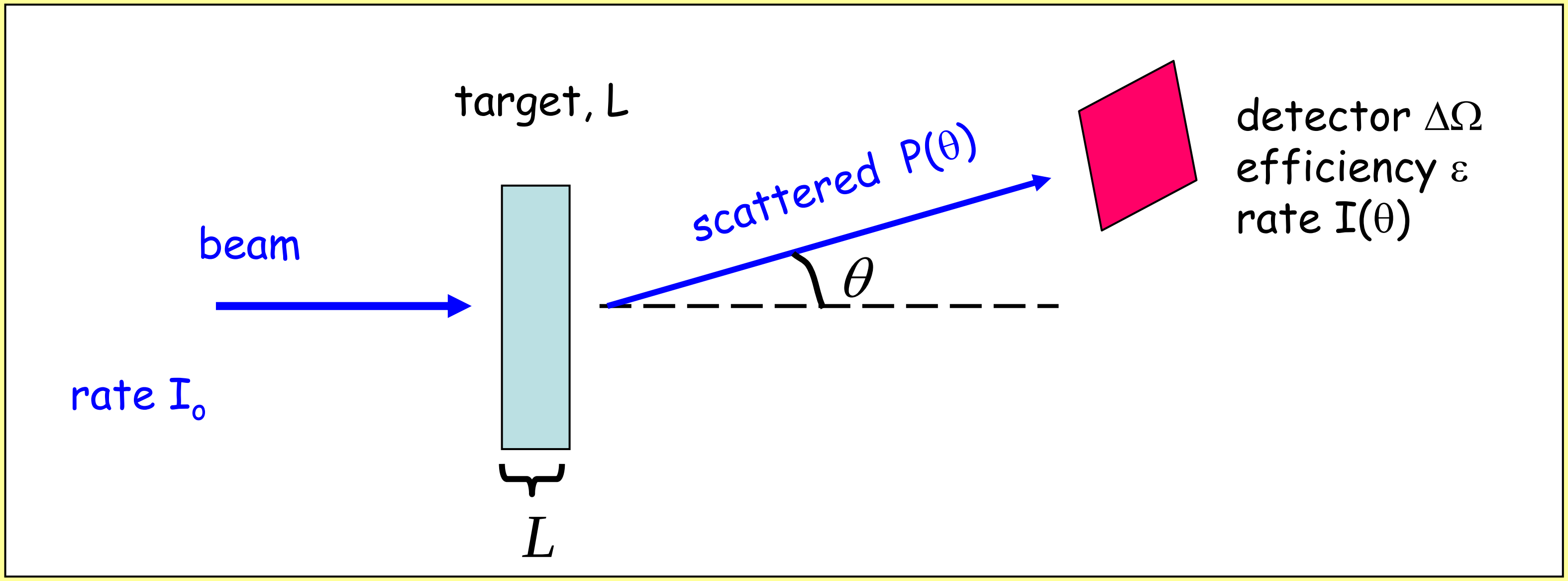
Otherwise, attenuation in the target has to be accounted for explicitly via the exponential relationship:

$$\boxed{P(x) \;\equiv\; 1 - e^{-n_t\, \sigma\, x}}$$

(always correct)

Thick target: $P(x) \to 1$, essentially independent of x beyond a certain thickness.

connection to experiment continued....

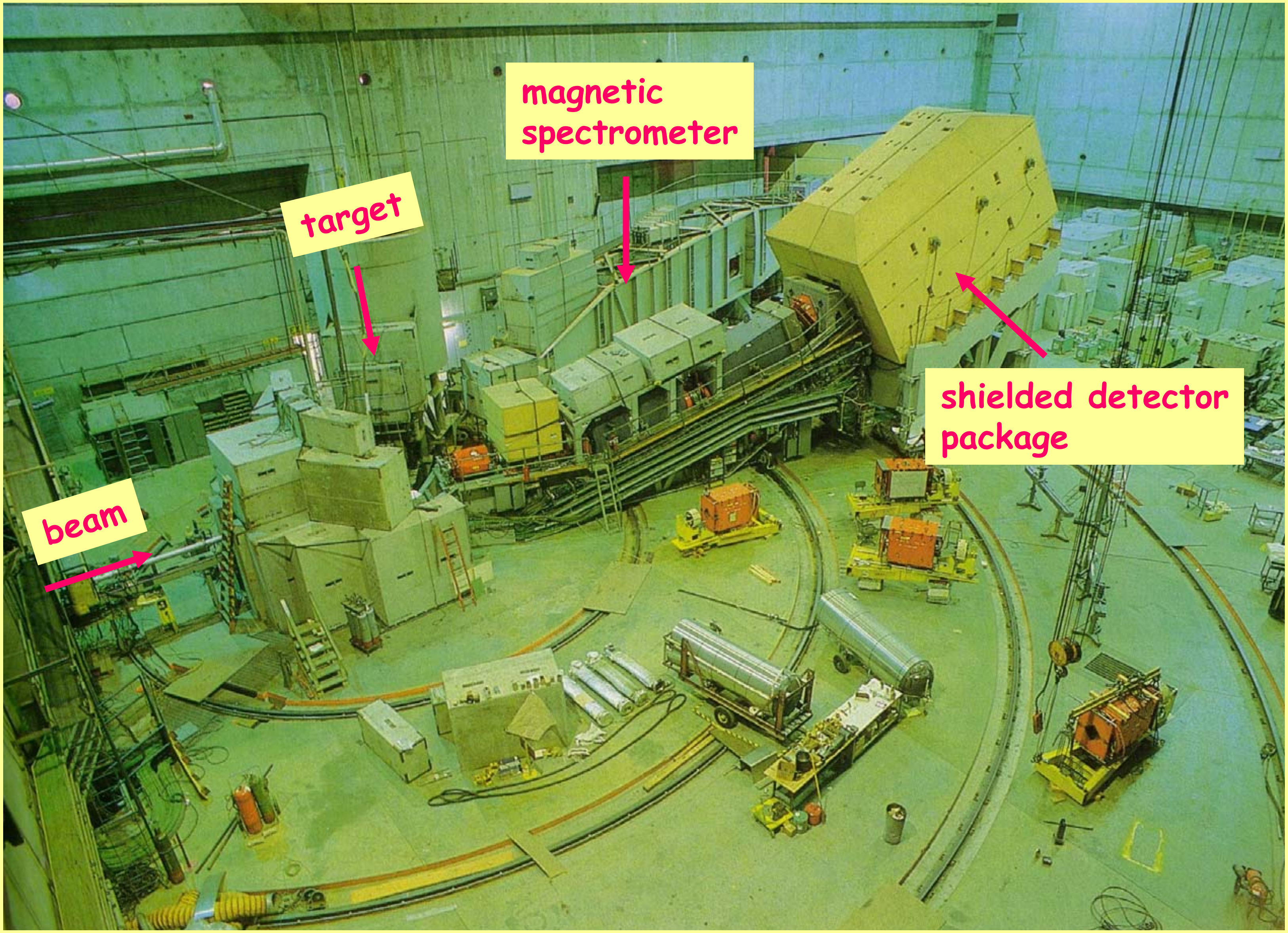


$$I(\theta) = I_o \; P(\theta, L) \; \varepsilon \; \Delta\Omega \;=\; I_o \; n_t \; L \left(\frac{d\sigma}{d\Omega}\right) \Delta\Omega \; \varepsilon \quad (small \quad \Delta\Omega)$$

(thin target case!)

$$= \; I_o \; n_t \; L \int_{\det} \varepsilon(\theta) \left(\frac{d\sigma(\theta)}{d\Omega}\right) \sin\theta \; d\theta \; d\phi$$

Electron scattering apparatus at SLAC:



Details, Part II: Statistical Accuracy in Experiments (*Ref: F&H sec. 4.2*)

Suppose we perform a scattering experiment for a certain time T. The differential cross section is determined from the ratio of scattered to incident beam particles in the same time period:

$$\boxed{N_o = I_o\, T, \quad N(\theta) = I(\theta)\, T \quad \Rightarrow \frac{d\sigma(\theta)}{d\Omega} = const.\ \frac{N(\theta)}{N_o}}$$

Scattering is a statistical, **random process**. Each beam particle will either scatter at angle θ or not, with probability $P(\theta)$. Individual scattering events are uncorrelated.

In this case, the statistical uncertainty in $N(\theta)$ is said to follow "**counting statistics**", and **the error in $N(\theta)$ determines the statistical uncertainty in $d\sigma/d\Omega$:**

$$\boxed{\frac{\sigma_N}{N} = \frac{1}{\sqrt{N}}}$$

(Note: strictly speaking, $N \gg 1$ for the Gaussian distribution to apply, but this is the usual case in a scattering experiment anyway.)

If we perform the same experiment many times, always counting for the same time T, we will measure many different values of N, the number of scattered particles… the distribution of values of N (where N >> 1) will be a **Gaussian or Normal distribution**, with the probability of observing a particular value given by:

$$P(N)\,dN = \frac{1}{\sigma_N\sqrt{2\pi}}\; e^{-(N-\bar{N})^2/2\sigma_N^2}\; dN; \qquad \int_0^{\infty} P(N)\,dN = 1$$

with standard derivation: $\boxed{\sigma_N = \sqrt{\bar{N}}}$ and mean value: $\boxed{\bar{N}}$

If we only do the measurement once, the best estimate of the statistical error comes from assuming that the distribution of events follows counting statistics as above.

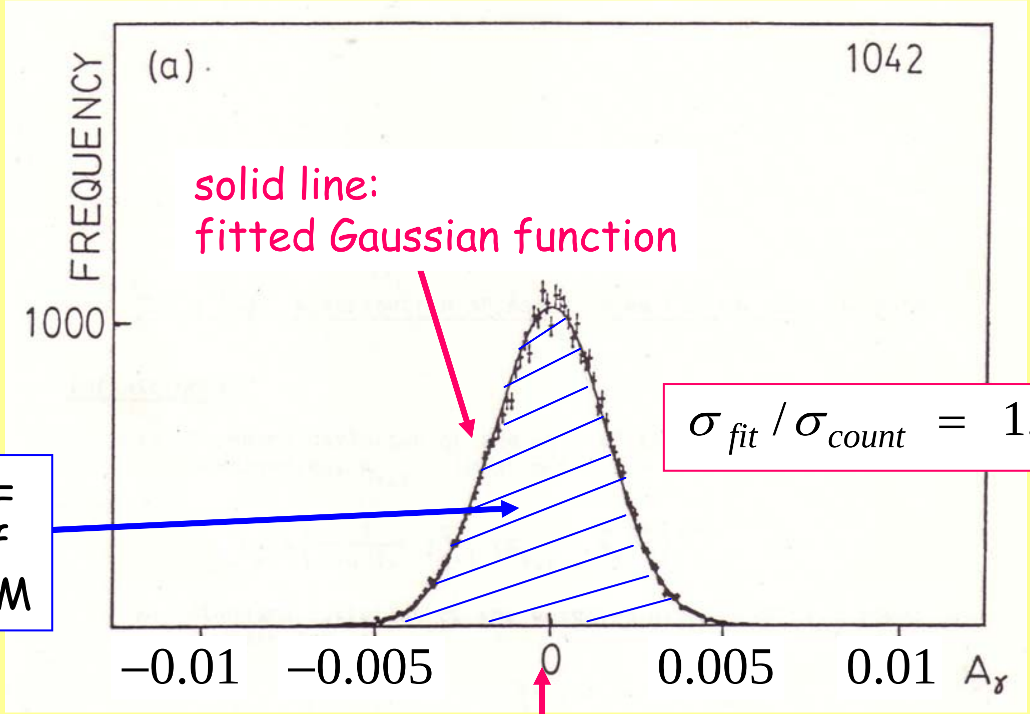
However, **it is important to verify that this is the case!**

*(Electrical noise, faulty equipment, computer errors etc. can lead to distributions of detected particles that do not follow counting statistics but in fact have much **worse** behavior. This will never do! ☹ …..)*

Example: checking on "counting statistics"

"Gamma Ray Asymmetry" histogram, Ph.D. thesis data (SAP);
Independent measurements of the number of gamma rays
N+ with the magnetic field "+" and N- with the field "-"

$$A_\gamma = \frac{N_+ - N_-}{N_+ + N_-}$$



$$\sigma_{fit} / \sigma_{count} = 1.001$$

Sum of entries = total number of measurements, M

Mean value: $A_\gamma = -(0.9 \pm 0.9) \times 10^{-5}$

$$\delta A_\gamma = \frac{\sigma_{fit}}{\sqrt{M}}$$

-- very good agreement with counting statistics based on values of N+ and N-

Notes:

1. Time T required to achieve a given statistical accuracy:

$$T \sim N \sim \left(\frac{\Delta\sigma}{\sigma}\right)^{-2}$$

2. Beam time is expensive, so nobody can afford to waste it!

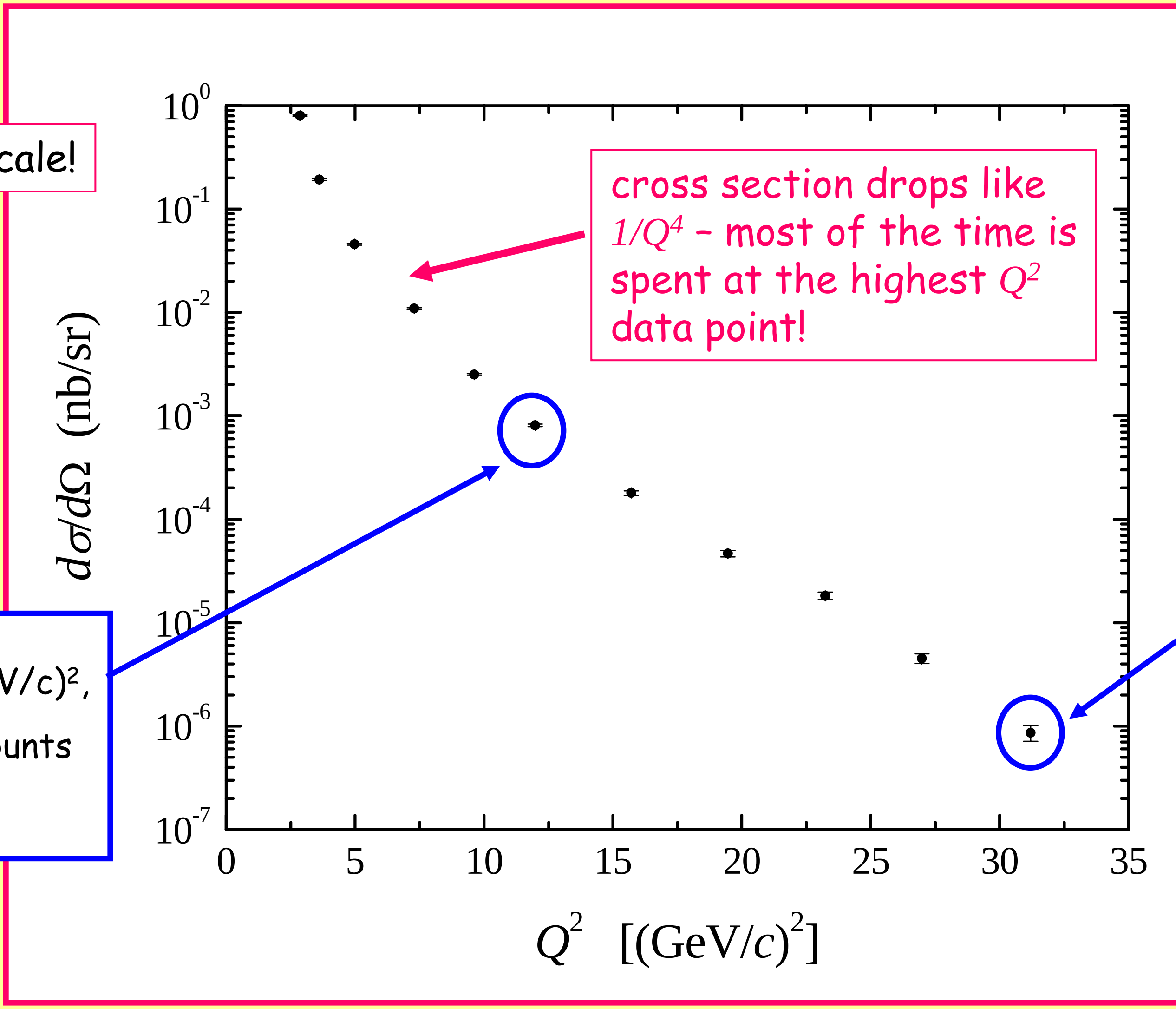
   e.g. at Jefferson Lab: 34 weeks/year x 2 beams costs US\$70M (lab budget)
   $\rightarrow$ \$625k Cdn/hour!

3. Efficient experiment design has statistical and systematic errors comparable, counting rate optimized for "worst" data point ($d\sigma/d\Omega$ smallest)

   $\rightarrow$ see example, next slide...

Data from High Energy Electron-Proton Scattering at SLAC:

Note log scale!



Q^2 = 12 (GeV/c)^2,
N = 1779 counts
$\Delta\sigma/\sigma$ = 2%

Q^2 = 31 (GeV/c)^2,
N = 39 counts
$\Delta\sigma/\sigma$ = 17%

(Statistical errors only shown)

Ref: Sill et al., Phys. Rev. D 48, 29 (1993)

Tables from Sill et al. SLAC experiment, 1993

TABLE V. Kinematic settings and elastic electron-proton cross sections. The cross-section results are the average of individual results given in the Appendix obtained from data taken under various beam and target conditions in parts I and II. Results at slightly different kinematics have been extrapolated to common values of E_0 and Q^2. The statistical and total systematic errors described in the text are listed separately.

E (GeV)	E' (GeV)	θ (deg)	Q^2 $[(\mathrm{GeV}/c)^2]$	$d\sigma/d\Omega \pm$stat$\pm$syst (nb/sr)
5.464	3.939	21.01	2.862	$0.802 \pm 0.009 \pm 0.029$
5.464	3.534	25.01	3.621	$0.193 \pm 0.004 \pm 0.007$
7.632	4.953	21.01	5.027	$(6.93 \pm 0.03 \pm 0.25) \times 10^{-2}$
6.657	3.998	25.01	4.991	$(4.55 \pm 0.08 \pm 0.16) \times 10^{-2}$
5.499	2.826	33.01	5.017	$(2.04 \pm 0.03 \pm 0.07) \times 10^{-2}$
9.606	5.716	21.01	7.300	$(1.09 \pm 0.02 \pm 0.04) \times 10^{-2}$
11.45	6.323	21.01	9.629	$(2.51 \pm 0.06 \pm 0.09) \times 10^{-3}$
13.21	6.824	21.01	11.99	$(8.08 \pm 0.21 \pm 0.29) \times 10^{-4}$
15.84	7.463	21.01	15.72	$(1.79 \pm 0.09 \pm 0.06) \times 10^{-4}$
18.36	7.979	21.01	19.47	$(4.67 \pm 0.32 \pm 0.17) \times 10^{-5}$
20.79	8.407	21.01	23.24	$(1.82 \pm 0.15 \pm 0.07) \times 10^{-5}$
21.18	6.796	25.01	26.99	$(4.51 \pm 0.50 \pm 0.16) \times 10^{-6}$
21.19	4.561	33.01	31.20	$(8.6 \pm 1.5 \pm 0.3) \times 10^{-7}$

Beam energy and spectrometer angle adjusted to vary the parameter of interest, momentum transfer Q^2

Independent systematic errors added in quadrature to the statistical error:

TABLE VIII. Sources of systematic uncertainty in $d\sigma/d\Omega$.

Source	$\Delta\sigma/\sigma$ (%)
Point to point uncertainties	
Incident energy	≤ 1.1
Scattering angle	0.5
Incident beam angle	1.0
Target density	0.5[a]
Beam charge	0.5
Radiative corrections	1.0
Sum in quadrature	2.0 (2.2)[a]
Overall uncertainties	
Optics	2.0
Acceptance normalization	2.0
Detector efficiencies	1.0
Sum in quadrature	3.0
Total	3.6 (3.7)[a]

[a]This uncertainty applies to the data point at $Q^2 = 31\ (\mathrm{GeV}/c)^2$ from target density variations due to beam heating in E136 I.